Comparing the Riskiness of Dependent Portfolios

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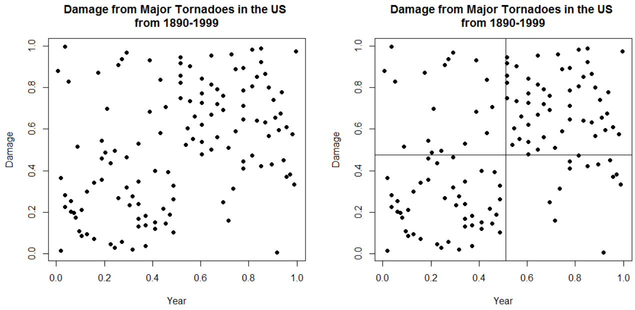
   - Motivation
   - Preliminaries
2. Hypothesis Test and Simulation Study
   - Nested L-Statistic
   - Asymptotic Results
   - Simulation Study
3. Conclusion

Motivation

- Data: The damage from 137 major tornadoes in the U.S. from 1890 to 1999: (see Brazauskas, Jones, Puri and Zitikis (2007))



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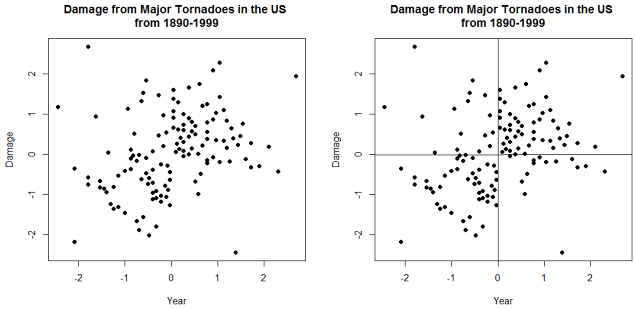

Motivation

- Motivation: If the risks are independent Brazauskas, Jones, Puri and Zitikis have introduced a hypothesis test to check if at least one of those is different from the others or not.

Questions:

1. How does this test behave in the presence of dependence?
2. Does this test perform in the same manner in the presence of different dependent structures?

Preliminaries

- Dependent portfolios:

Negative Dependence:

$$\Sigma = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix}$$

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Strong Positive Dependence:

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Preliminaries

- Spectral risk measure:

$$R[F] = \int_0^1 F^{-1}(u)J(u)\,\mathrm{d}u$$

where J is such that the integral is finite for the set of cdf's F under consideration (Jones & Zitikis, 2003). J is called a *risk aversion function*.

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- Examples:
  - MEAN:

$$J(u) = 1 \quad \text{for} \quad 0 \le u \le 1$$

Preliminaries

- Proportional Hazards Transform (PHT)

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- Conditional Tail Expectation (CTE)

$$J(u) = \begin{cases} 0, & \text{for } \; 0 \le u < t, \\ 1/(1-t), & \text{for } \; t \le u \le 1. \end{cases}$$

where t $(0 \le t < 1)$ is called the *threshold level*. Alternative names for the CTE are: *Tail Conditional Expectation*; *Conditional Value-at-Risk*; *Expected Shortfall*.

Preliminaries

- Nonparametric estimation of risk measures:

$$\widehat{R[F]} = R[\widehat{F}] = \int_0^1 \widehat{F}^{-1}(u) J(u)\,\mathrm{d}u = \sum_{m=1}^{n} \left(\int_{(m-1)/n}^{m/n} J(u)\,\mathrm{d}u \right) X_{m:n} = \sum_{m=1}^{n} c_{nm} X_{m:n}$$

where $\widehat{F}$ is the empirical cdf based on the sample $X_1, \ldots, X_n$ with $X_{1:n} \leq \cdots \leq X_{n:n}$ denoting its order statistics.
Note: $\widehat{R[F]}$ is an *L-statistic* (i.e., a linear combination of order statistics).

Preliminaries

- Hypothesis test:

Let $R_1 = R[F_1], \ldots, R_k = R[F_k]$ be risk measure values corresponding to k populations with cdf's $F_1, \ldots, F_k$ which can be *dependent* or *independent*. The hypothesis of interest:

$$H_0: \quad R_1 = \cdots = R_k, \quad \text{Vs.}$$
$$H_1: \quad \text{for at least one pair } (i, j), \quad R_i \neq R_j.$$

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- The reformulation of the test using Gini index

$$H_0 : \quad \gamma = 0 \quad \text{Vs.} \quad H_1 : \quad \gamma > 0$$

where $\gamma := k^{-2} \sum_{1 \le i,j \le k} |R_i - R_j|$ is the Gini index (Gini, 1914) of the risk measure values $R_1, \ldots, R_k$.

Nested L-Statistic

- The Gini index γ as a Nested-L statistic:

$$\gamma = \frac{1}{k^2} \sum_{1 \le i,j \le k} |R_i - R_j| = \frac{1}{k^2} \sum_{i=1}^{k} (4i - 2(k+1)) R_{i:k}$$

$$= \sum_{i=1}^{k} \left(\int_{(i-1)/k}^{i/k} K(u) du \right) R_{i:k} = \sum_{i=1}^{k} c^*_{ki} R_{i:k}$$

where $K(u) := 4u - 2$ and $R_{1:k} \le \cdots \le R_{k:k}$ are the k ordered risk measure values.

Nested L-Statistic

- A nonparametric estimator of γ:

$$\begin{aligned}\widehat{\gamma} &= \frac{1}{k^2}\sum_{1\le i,j\le k}\left|\widehat{R}_i-\widehat{R}_j\right| \\ &= \frac{1}{k^2}\sum_{i=1}^{k}\left(4i-2(k+1)\right)\widehat{R}_{i:k} \\ &= \sum_{i=1}^{k} c^*_{ki}\widehat{R}_{i:k}\end{aligned}$$

where $\widehat{R}_i=\sum_{m=1}^{n} c_{nm}X_{m:n}(i)$ and $\widehat{R}_{1:k}\le\cdots\le\widehat{R}_{k:k}$ denote the k ordered *estimators* of the corresponding risk measure values.

Asymptotic Results

- The test statistic:

$$T := \sqrt{\frac{n}{k}}\widehat{\gamma}$$

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Then under H_0,

$$T = \frac{1}{k^2}\sum_{i=1}^{k}\left(4i - 2(k+1)\right)\Delta_{i:k} \tag{1}$$

where $\Delta_i = \sqrt{\frac{n}{k}}\left(\widehat{R_i} - R_i\right)$ and $\Delta_{1:k} \leq \cdots \leq \Delta_{k:k}$.

Asymptotic Results

- Under H_0 :
  As $n \to \infty$, the asymptotic distribution of T is

$$\frac{1}{k^2}\sum_{i=1}^{k}(4i - 2(k+1))\, G_{i:k}$$

where, $G_{1:k}, \ldots, G_{k:k}$ are k order statistics of *normal* random variables with the same mean $(= 0)$ but with different (and "messy") variances. Their dependence depend on that of underlying risks.

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- Under H_1 :
  As $n \to \infty$, the test statistic $T \to \infty$, implying that the asymptotic power of the test is 1.

Asymptotic Results

- Bootstrap:
  - For every $1 \leq i \leq k$, resample (with replacement) $X_1(i), \ldots, X_n(i)$ and obtain $X_1^*(i), \ldots, X_n^*(i)$; then compute

$$\widehat{\gamma}^* := \frac{1}{k^2} \sum_{i=1}^{k} (4i - 2(k+1))\, D_{i:k}^*$$

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  - Repeat the previous step B (say, $B = 1000$) number of times and obtain $\widehat{\gamma}_1^*, \ldots, \widehat{\gamma}_B^*$; then order them and obtain $\widehat{\gamma}_{1:B}^* \leq \cdots \leq \widehat{\gamma}_{B:B}^*$.

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  - DECISION: Reject H_0 at the α level, if $\widehat{\gamma} > \widehat{\gamma}_{[B(1-\alpha)]:B}^*$

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- Risk measures (R's):

  MEAN, $\text{PHT}\,[r = 0.85]$, $\text{CTE}\,[t = 0.75]$

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- Under H_0 (equally risky portfolios)

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- Under H_1 (unequally risky portfolios)

  For a fixed R, we consider two types of alternatives:

  1- Two portfolios are equally risky but the third one differs:

$$R[F_1^\star] = c_\star R[F_1], \quad R[F_2^\star] = R[F_2], \quad R[F_3^\star] = R[F_3],$$

  where $R[F_1] = R[F_2] = R[F_3]$ and $c_\star \neq 1$.

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where $R[F_1] = R[F_2] = R[F_3]$ and $c_\star \neq 1$.

2- Relative riskiness of all three portfolios is equally-spaced:

$$R[F_1^{\star\star}] = c_{\star\star} R[F_1], \quad R[F_2^{\star\star}] = R[F_2], \quad R[F_3^{\star\star}] = c_{\star\star}^2 R[F_3],$$

where $R[F_1] = R[F_2] = R[F_3]$ and $c_{\star\star} > 1$.

Simulation study

TABLE 1: Estimated power of the tests for various dependence structures based on the MEAN and PHT, measures, for $n = 200$ and $\alpha = 0.05$.

Risk Measure	Dependence	Alternate 1-constants ($c_\star$)									
		0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.25	1.50	2.00
MEAN	Negative	1.00	1.00	0.65	0.05	0.47	0.94	0.98	1.00	1.00	1.00
	Independence	1.00	1.00	0.82	0.05	0.59	0.98	1.00	1.00	1.00	1.00
	Moderate-Positive	1.00	1.00	0.94	0.06	0.90	0.99	1.00	1.00	1.00	1.00
	Strong-Positive	1.00	1.00	0.98	0.06	0.99	1.00	1.00	1.00	1.00	1.00
PHT	Negative	1.00	1.00	0.48	0.05	0.43	0.87	0.97	1.00	1.00	1.00
	Independence	1.00	1.00	0.55	0.05	0.51	0.93	0.99	1.00	1.00	1.00
	Moderate-Positive	1.00	1.00	0.83	0.05	0.74	0.97	1.00	1.00	1.00	1.00
	Strong-Positive	1.00	1.00	0.90	0.06	0.89	0.99	1.00	1.00	1.00	1.00

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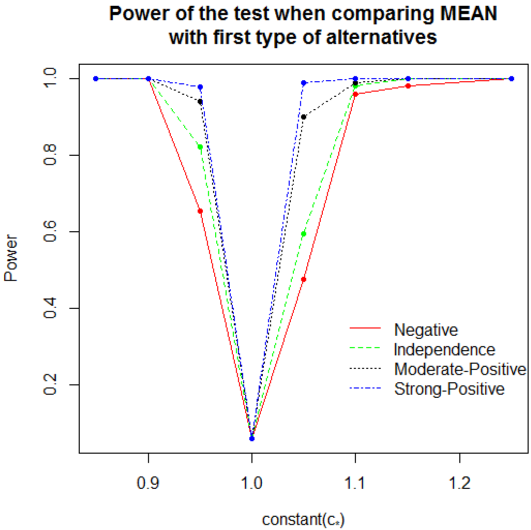

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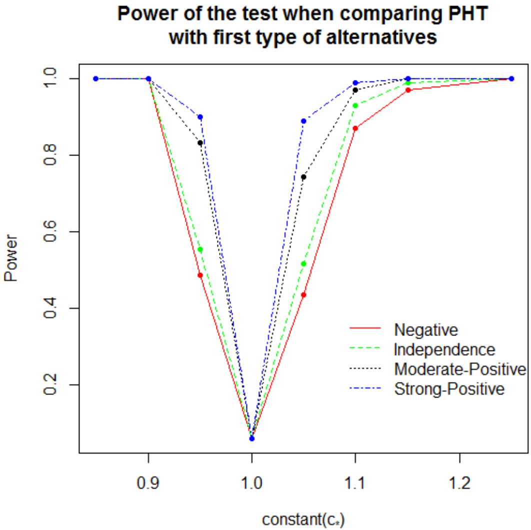

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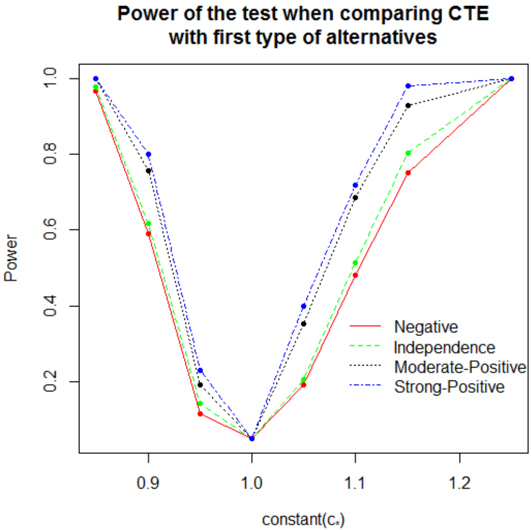

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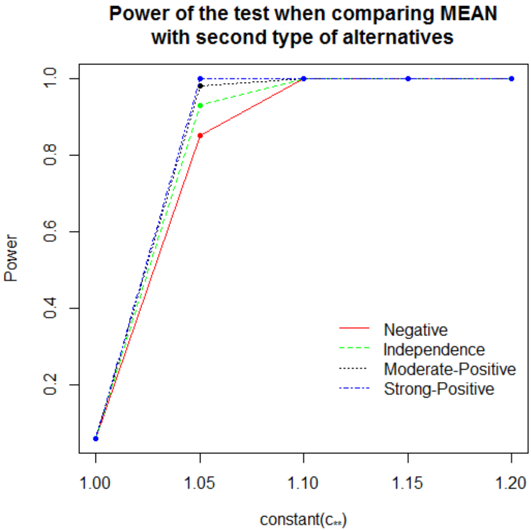

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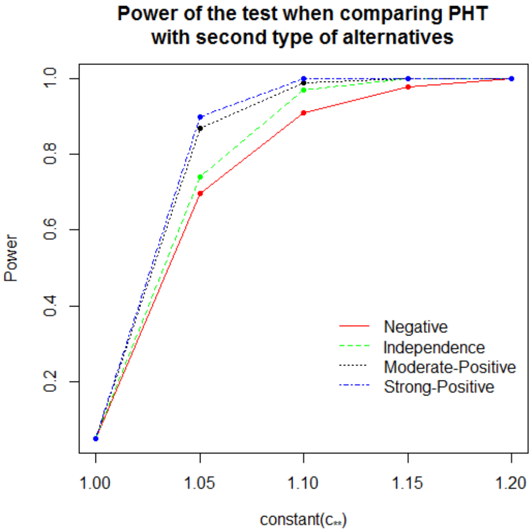

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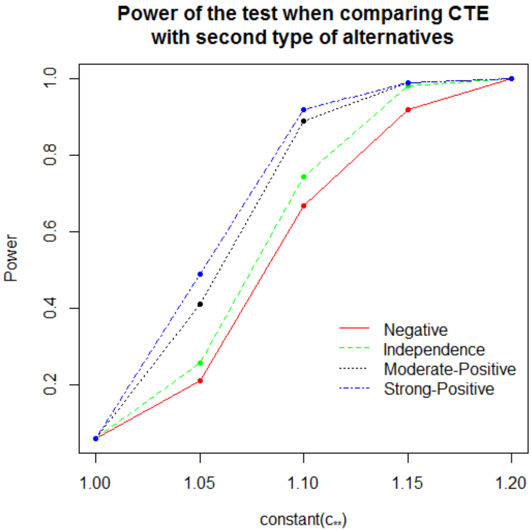

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2. The presence of negative dependence among the portfolios makes the test less powerful for the risk measures under consideration.